Contents

1 Introduction to Inequalities 1
   1.1 Law of trichotomy ........................................ 1
   1.2 Basic properties of inequalities .......................... 1
   1.3 Other elementary properties of inequalities ............ 1
   1.4 AM-GM inequality ........................................... 3
   1.5 Sides of a triangle .......................................... 3

2 Classical inequalities 5
   2.1 Mean inequalities ........................................... 5
   2.2 Generalized mean inequality (or power mean inequality) .... 5
   2.3 Jensen’s inequality ........................................ 5
   2.4 Cauchy–Schwarz inequality ............................... 6
   2.5 Hölder’s inequality ......................................... 6
   2.6 Rearrangement inequality .................................. 6
   2.7 Chebeshev’s sum inequality ................................. 6
   2.8 Minkowski’s inequality ...................................... 6
   2.9 Majorization of finite sequences .......................... 7
   2.10 Karamata’s inequality ..................................... 7
   2.11 Cyclic and symmetric summation .......................... 7
   2.12 Muirhead’s inequality ...................................... 7
   2.13 Schur’s inequality .......................................... 8
   2.14 Elementary symmetric polynomial ....................... 8
   2.15 Symmetric mean inequalities ............................. 8
   2.16 Bernoulli’s inequality ...................................... 8

3 Examples 9

Notes on Inequalities
1 Introduction to Inequalities

1.1 Law of trichotomy
For real numbers $x$ and $y$, exactly one of the following holds: $x < y$, $x = y$, $x > y$.

1.2 Basic properties of inequalities
Here are the basic properties of inequalities, which are introduced in secondary schools:

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitive property</td>
<td>If $a &gt; b$, $b &gt; c$, then $a &gt; c$. Hence we write $a &gt; b &gt; c$.</td>
</tr>
<tr>
<td>Additive property</td>
<td>If $a &gt; b$, then $a + c &gt; b + c$ for all real number $c$.</td>
</tr>
<tr>
<td>Multiplicative property</td>
<td>If $a &gt; b$, $c &gt; 0$, then $ac &gt; bc$.</td>
</tr>
<tr>
<td></td>
<td>If $a &gt; b$, $c &lt; 0$, then $ac &lt; bc$.</td>
</tr>
<tr>
<td>Reciprocal property</td>
<td>If $a &gt; b &gt; 0$ or $0 &gt; a &gt; b$, then $1/a &lt; 1/b$.</td>
</tr>
</tbody>
</table>

1.3 Other elementary properties of inequalities
Besides the basic properties, here are a few other useful properties on inequalities:

(a) $a^2 > 0$ for all real number $a$;
(b) If $a > b$ and $c > d$, then $a + c > b + d$ and $a - d > b - c$;
(c) If $a > b > 0$ and $c > d > 0$, then $ac > bd$;
(d) If $a > b > 0$ and $0 > c > d$, then $ad < bc$;
(e) If $0 < a < 1 < b$ and $k > 0$, then $0 < a^k < 1 < a^{-k}$ and $0 < b^{-k} < 1 < b^k$;
(f) If $0 < a < b$ and $k > 0$, then $a^k < b^k$ and $a^{-k} > b^{-k}$.

Example: Show that $x^2 + y^2 \geq 2xy$, where $x, y$ are real numbers.
Solution: $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$, which reduces to the given inequality. Equality case holds when $x = y$.

Example: Show that $(a + b)(b + c)(c + a) \geq 8abc$, where $a, b, c$ are non-negative real numbers. Determine where equality case holds.
Solution 1: We have $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $c + a \geq 2\sqrt{ca}$. Multiplying all three inequality together, we get the required inequality. Equality case holds when $x = y = z$, $x = y = 0$, $y = z = 0$ or $z = x = 0$.
Solution 2: The inequality can be reduced into $a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 \geq 6abc$. This can be further reduced into $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0$, which is true. (For equality cases, see solution 1.)

Notes on Inequalities
Example: (IMO 1969 Longlisted (HUN)) Prove that \(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}\).

Solution: Note that \(\frac{1}{n^3} < \frac{1}{n(n-1)(n+1)} = \frac{1}{2} \left( \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \right)\). Therefore,

\[
1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} = 1 + \sum_{i=2}^{n} \frac{1}{i^3} < 1 + \sum_{i=2}^{n} \frac{1}{2} \left( \frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1} \right)
= 1 + \frac{1}{2} \left( \frac{1}{1} - \frac{1}{2} - \frac{1}{n} + \frac{1}{n+1} \right) < 1 + \frac{1}{2} \left( \frac{1}{1} - \frac{1}{2} \right) = \frac{5}{4} \quad \Box
\]

(Note: when \(n\) approaches infinity, the result is known as the Apéry’s constant. Its value is approximately equal to 1.202.)

Exercise:

1. Show that \(a^2 + b^2 + c^2 \geq ab + bc + ca\), where \(a, b, c\) are real numbers. Determine when does equality hold.

2. Prove that \(\sqrt{k+1} + \sqrt{k-1} < 2\sqrt{k}\) and \(\frac{1}{\sqrt{k}} < \sqrt{k+1} - \sqrt{k-1}\), where \(k > 1\).

3. Prove that \(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < \sqrt{n+1} + \sqrt{n-1}\) for any positive integer \(n\).

4. Show that \(\left( \frac{n+1}{2} \right)^n > n!\) for any integer \(n > 1\).

5. (IMO 1970 Longlisted (FRA)) Let \(n\) and \(p\) be two integers such that \(2p \leq n\). Prove the inequality

\[
\frac{(n-p)!}{p!} \leq \left( \frac{n+1}{2} \right)^{n-2p}.
\]

6. (IMO 1975 Shortlisted (SWE)) Let \(M\) be the set of all positive integers that do not contain the digit 9 (base 10). If \(x_1, \ldots, x_n\) are arbitrary but distinct elements in \(M\), prove that

\[
\sum_{j=1}^{n} \frac{1}{x_j} < 80.
\]

7. (IMO 1982 #3) Consider the infinite sequences \(\{x_n\}\) of positive real numbers with the following properties: \(x_0 = 1\) and for all \(i \geq 0\), \(x_{i+1} \leq x_i\).

(a) Prove that for every such sequence there is an \(n \geq 1\) such that

\[
\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.
\]

(b) Find such a sequence for which \(\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4\) for all \(n\).

Prepared by Leung W.C.
1.4 AM-GM inequality

For non-negative real numbers $a_1, a_2, \ldots, a_n$, we have

\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},
\]

where the equality case happen if and only if $a_1 = \cdots = a_n$.

The expression $\frac{a_1 + a_2 + \cdots + a_n}{n}$ is known as the arithmetic mean (AM) of the $n$ numbers, and the expression $\sqrt[n]{a_1 a_2 \cdots a_n}$ is known as the geometric mean (GM) of the $n$ numbers.

Example: Given $a_1, \ldots, a_n \geq 0$. Show that $\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$.

Solution: By AM-GM inequality, we have $\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}}$, which reduces to the required inequality. The equality case hold when $a_1 = \cdots = a_n$. \hfill \square

Note: the right hand side of the inequality is known as the harmonic mean (HM).

Example:

Show that if $a$, $b$, $c$ are positive real numbers, then $a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Solution: By AM-GM inequality, we have

\[
2a^4 + b^4 + c^4 \geq 4a^2bc \\
a^4 + 2b^4 + c^4 \geq 4ab^2c \\
a^4 + b^4 + 2c^4 \geq 4abc^2
\]

Adding up becomes $4(a^4 + b^4 + c^4) \geq 4abc(a + b + c)$, which reduces to the required inequality. Equality case holds when $a = b = c$. \hfill \square

1.5 Sides of a triangle

Line segments with lengths $a$, $b$, $c$ can form a triangle if and only if $a + b > c$, $b + c > a$ and $c + a > b$. This is known as the triangle inequality.

Also, if $a$, $b$, $c$ are the lengths of the sides of a triangle, we can make the substitution $x = \frac{c+a-b}{2}$, $y = \frac{a+b-c}{2}$ and $z = \frac{b+c-a}{2}$. By triangle inequality, we have $x, y, z > 0$. Solving for $a$, $b$, and $c$, we obtain $a = x + y$, $b = y + z$, $c = z + x$ and $a + b + c = 2(x + y + z)$.

Of course, other formulas related to triangles can also be useful, i.e. $S = \frac{1}{2}ab \sin C$, cosine formula ($c^2 = a^2 + b^2 - 2ab \cos C$) and Heron’s formula ($S = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$).
**Example: (IMO 1964 #2)** Let $a$, $b$, $c$ be the lengths of the sides of a triangle. Prove that $a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$.

Solution: Let $x = \frac{c+a-b}{2}$, $y = \frac{a+b-c}{2}$ and $z = \frac{b+c-a}{2}$. We have $a = x + y$, $b = y + z$, $c = z + x$. Now the given inequality is equivalent to

$$(x + y)^2(2z) + (y + z)^2(2x) + (z + x)^2(2y) \leq 3(x + y)(y + z)(z + x)$$

$\iff$

$$2 \left( \frac{x^2y + xy^2 + y^2z + yz^2}{yz^2 + z^2x + zx^2} \right) + 12xyz \leq 3 \left( \frac{x^2y + xy^2 + y^2z + yz^2}{yz^2 + z^2x + zx^2} \right) + 6xyz$$

$\iff$

$$6xyz \leq x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2$$

Which is a direct result of AM-GM inequality. Since $x, y, z > 0$, the equality case of the above inequality is $x = y = z$, i.e. $a = b = c$. $\square$

**Exercise:**

1. Given $x, y, z \geq 0$. Prove that $x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z)$.
   (Does the equality hold also for all real numbers $x, y$ and $z$?)

2. Let $a$, $b$, $c$ be the lengths of the sides of a triangle. Show that $(a + b + c)(b + c - a) < 4bc$.

3. Let $a$, $b$, $c$ be real numbers. If $a + b + c = 1$, prove that $a^2 + b^2 + c^2 \geq \frac{1}{3}$.

4. Given $-1 < x, y, z < 1$. Prove that

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2.$$ 

5. (BMO 2001) Let $a$, $b$, $c$ be positive real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

6. (Yugoslavia 1987) Given $a, b > 0$. Prove that $\frac{1}{2}(a + b)^2 + \frac{1}{4}(a + b) \geq a\sqrt{b} + b\sqrt{a}$.

7. (Nesbitt’s inequality) Given $a, b, c > 0$. Prove that $\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}$.

8. (IMO 1969 Longlisted (YUG))
   Suppose that positive real numbers $x_1, x_2, x_3$ satisfy $x_1x_2x_3 > 1$, $x_1 + x_2 + x_3 < \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$.
   Prove that:

   (a) None of $x_1, x_2, x_3$ equals 1.

   (b) Exactly one of these numbers is less than 1.

Prepared by Leung W.C.
2 Classical inequalities

2.1 Mean inequalities

For positive real numbers $a_1, \ldots, a_n$, we have
\[ QM = \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}, \quad AM = \frac{a_1 + \cdots + a_n}{n}, \]
\[ GM = \sqrt[n]{a_1 a_2 \cdots a_n}, \quad HM = \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}, \]
the equality case is $a_1 = \cdots = a_n$. $QM$, $AM$, $GM$ and $HM$ stands for quadratic mean, arithmetic mean, geometric mean and harmonic mean respectively. If $HM$ is not involved, the conditions can be relaxed into $a_1, \ldots, a_n \geq 0$.

2.2 Generalized mean inequality (or power mean inequality)

For positive real numbers $a_1, \ldots, a_n$, we define the generalized mean or power mean with exponent $p$ as follows:
\[ M_p = \begin{cases} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} & \text{for } p \neq 0 \\ \lim_{p \to 0} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} = \sqrt[n]{a_1 a_2 \cdots a_n} & \text{for } p = 0 \end{cases} \]
If $p > q$, then we have $M_p \geq M_q$, with the equality case $a_1 = \cdots = a_n$.

Note the following special cases: $M_{+\infty} = \max(a_1, \ldots, a_n)$, $M_2 = QM$, $M_1 = AM$, $M_0 = GM$, $M_{-1} = HM$, $M_{-\infty} = \min(a_1, \ldots, a_n)$.

2.3 Jensen’s inequality

For a convex function $f$, numbers $x_1, x_2, \ldots, x_n$ in its domain, and positive weights $a_1, \ldots, a_n$ that $a_1 + \cdots + a_n = 1$, Jensen’s inequality can be stated as
\[ f\left( \sum_{i=1}^{n} a_i x_i \right) \leq \sum_{i=1}^{n} a_i f(x_i). \]
If the weights are equal, then we have $f\left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i)$.

For a concave function, the inequality sign is reversed.

Note: function $f$ is convex if, for any points $x_1, x_2$ is in the domain of $f$, $t \in (0, 1)$, we have $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ provided that $f(tx_1 + (1-t)x_2)$ is defined. This is the two variable case of Jensen’s inequality.
2.4 Cauchy–Schwarz inequality

For real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right).$$

Equality holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ or $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$.

2.5 Hölder’s inequality

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers. If $p, q > 0$, $1/p + 1/q = 1$, we have

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Equality holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ or $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$. The CS inequality is a special case of Hölder’s inequality where $p = q = 2$.

2.6 Rearrangement inequality

Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation $\sigma$ of $\{1, \ldots, n\}$, we have the following:

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma_i} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

2.7 Chebeshev’s sum inequality

Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have the following:

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i\right) \left(\sum_{i=1}^{n} y_i\right) \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

2.8 Minkowski’s inequality

For non-negative real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, $p \geq 1$, we have

$$\left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} x_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} y_k^p\right)^{1/p}.$$

Equality case holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$, $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$, or $p = 1$.

When $p = 2$, then we obtain the triangle inequality in the $n$-dimensional space:

$$\sqrt{n} \left(\sum_{k=1}^{n} (x_k + y_k)^2\right)^{1/2} \leq \sqrt{n} \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \sqrt{n} \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}.$$

Prepared by Leung W.C.
2.9 Majorization of finite sequences

Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) be two sequences of real numbers such that \( a_1 \geq a_2 \geq \cdots \geq a_n \) and \( b_1 \geq b_2 \geq \cdots \geq b_n \). Sequence \( a \) majorizes sequence \( b \) if the following two conditions are satisfied:

(i) \( a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k \), for all \( k \) where \( 1 \leq k \leq n - 1 \);

(ii) \( a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n \).

We write \( a \succ b \) or \( b \prec a \) if \( a \) majorizes \( b \).

2.10 Karamata’s inequality

For a convex function \( f \), given \((x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)\), we have

\[
f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).
\]

The inequality sign is reversed if the function is concave.

Karamata’s inequality is also known as majorization inequality. Note that it is a generalization of Jensen’s inequality.

2.11 Cyclic and symmetric summation

For inequalities we often have to do summation in cyclic and symmetric way. To avoid writing a lot of terms, we use the symbol \( \sum_{\text{cyc}} \) for cyclic sums, and \( \sum_{\text{sym}} \) for symmetric sums, i.e. summation of all terms in the permutation. For example:

\[
\sum_{\text{cyc}(x,y,z)} x^2y = x^2y + y^2z + z^2x \\
\sum_{\text{sym}(x,y,z)} x^2y = x^2y + y^2z + z^2x + y^2x + z^2y + x^2z
\]

\( \sum_{\text{cyc}(x,y,z)} x^2y \) can be notated \( \sum_{x,y,z} x^2y \), while \( \sum_{\text{sym}(x,y,z)} x^2y \) can be notated \( \sum_{\text{sym}} x^2y^1z^0 \).

2.12 Muirhead’s inequality

Let \( x_1, \ldots, x_n \) be non-negative integers. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \succ \beta = (\beta_1, \ldots, \beta_n) \), then

\[
\sum_{\text{sym}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq \sum_{\text{sym}} x_1^{\beta_1} \cdots x_n^{\beta_n}
\]

Equality case holds if \( \alpha = \beta \) or \( x_1 = \cdots = x_m \). Conversely, if the inequality holds for all non-negative \( x_1, \ldots, x_n \), then \( \alpha \succ \beta \).
2.13 Schur’s inequality

For positive numbers \(x, y, z\) and real number \(t\), we have
\[
x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \geq 0
\]
Equality holds if \(x = y = z\) or two of them are equal and the third is zero. If \(t\) is a positive even number, then the inequality holds for all real numbers \(x, y, z\). If \(t\) is positive, then the inequality holds for all non-negative numbers \(x, y, z\).

Schur’s inequality can be used to form many difficult to prove inequalities.

When \(r = 1\), we have:
\[
a^3 + b^3 + c^3 + 3abc \geq a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2
\]
\[
abc \geq (a + b - c)(b + c - a)(c + a - b)
\]
When \(r = 2\), we have:
\[
a^4 + b^4 + c^4 + abc(a + b + c) \geq a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3
\]

2.14 Elementary symmetric polynomial

Consider the coefficients \(e_i\) of polynomial \((t + x_1) \cdots (t + x_n) = t^n + e_1 t^{n-1} + \cdots + e_{n-1} t + e_n\).

For example:
\[
e_1(x, y, z) = x + y + z
\]
\[
e_2(x, y, z) = xy + yz + zx
\]
\[
e_3(x, y, z) = xyz
\]

The coefficients \(e_1, \ldots, e_n\) are known as the elementary symmetric polynomials.

2.15 Symmetric mean inequalities

We define \(S_i = e_i/\binom{n}{i}\). \(S_i\) are known as the symmetric means. Note that \(S_1\) is the arithmetic mean, and \(S_n\) is the geometric mean. We have the following inequalities:

Newton’s inequality: \(S_i^2 \geq S_{i+1} S_{i-1}\), and

Maclaurin’s inequality: \(S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n}\).

2.16 Bernoulli’s inequality

For all integer \(r \geq 1\) and \(x \geq -1\), we have \((1 + x)^r \geq 1 + rx\). It can be generalized to real exponents: for \(x > -1, x \neq 0\), we have
\[
\begin{cases}
(1 + x)^a > 1 + ax & \text{for } a > 1 \text{ or } a < 0 \\
(1 + x)^a < 1 + ax & \text{for } 0 < a < 1
\end{cases}
\]
Note that equality holds if \(x = 0\).
3 Examples

Example 1: (IMO 1970 Longlisted (AUT)) Prove that
\[ \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leq \frac{1}{2}(a + b + c) \quad (a, b, c > 0). \]

Solution: By GM-HM inequality,
\[ \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} = \frac{1}{\frac{1}{b} + \frac{1}{c}} + \frac{1}{\frac{1}{c} + \frac{1}{a}} + \frac{1}{\frac{1}{a} + \frac{1}{b}} \]
\[ < \frac{1}{2} \left( \frac{b + c}{2} + \frac{c + a}{2} + \frac{a + b}{2} \right) \]
\[ = \frac{1}{2}(a + b + c). \]

Equality case holds for \( a = b = c \).

Example 2: Let \( a, b, c \geq 0 \). Prove that \( a^8 + b^8 + c^8 \geq a^6b^2c^2 + a^6c^2b^2 + b^6c^2a^2 \).

Solution 1: We have \((8, -1, -1) \succ (6, 0, 0)\). Therefore the given inequality is a result of Muirhead’s inequality. Equality case holds when \( a = b = c \).

(Note: The given inequality can be rewritten as \( \frac{1}{2} \sum_{\text{sym}} a^3 b^{-1} c^{-1} \geq \frac{1}{2} \sum_{\text{sym}} a^6 b^0 c^0 \).)

Solution 2: The given inequality is equivalent to \( a^9 + b^9 + c^9 \geq abc(a^6 + b^6 + c^6) \).

WLOG, assume \( a \geq b \geq c \). By Chebyshev’s sum inequality and AM-GM inequality, we have
\[ a^9 + b^9 + c^9 \geq \frac{1}{3}(a^3 + b^3 + c^3)(a^6 + b^6 + c^6) \geq abc(a^6 + b^6 + c^6), \]
which completes the proof. Equality case holds when \( a = b = c \).

Example 3: Let \( a, b, c \geq 0 \). Prove that \( a^5 + b^5 + c^5 \geq a^4b + b^4c + c^4a \).

Solution: If \( a \geq b \geq c \), we have \( a^4 \geq b^4 \geq c^4 \). If \( a, b \) and \( c \) has a different order, we have a similar result. By rearrangement inequality, we have
\[ a^5 + b^5 + c^5 = a^4 \cdot a + b^4 \cdot b + c^4 \cdot c \geq a^4b + b^4c + c^4a. \]

Equality case holds when \( a = b = c \).

(Note: avoid using the phrase ”WLOG, assume \( a \geq b \geq c \)” in this question because the expression \( a^4b + b^4c + c^4a \) is not symmetric.)
**Example 4: (IMO 1966 Longlisted (CZS))**

(a) Prove that \((a_1 + a_2 + \cdots + a_k)^2 \leq k (a_1^2 + \cdots + a_k^2)\), where \(k \geq 1\) is a natural number and \(a_1, \ldots, a_k\) are arbitrary real numbers.

(b) If real numbers \(a_1, \ldots, a_n\) satisfy

\[a_1 + a_2 + \cdots + a_n \geq \sqrt{(n-1)(a_1^2 + \cdots + a_n^2)},\]

show that they are all nonnegative.

**Solution:**

(a) Let \(b_1 = \cdots = b_k = 1\). We have

\[(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k)^2 \leq (a_1^2 + \cdots + a_k^2) (b_1^2 + \cdots + b_k^2),\]

which reduces to the required inequality. Equality case holds when \(a_1 = \cdots = a_k\).

(b) WLOG, assume that \(a_1, \ldots, a_j\) are negative, where \(1 \leq j \leq n\). Now

\[a_1 + a_2 + \cdots + a_n < a_{j+1} + \cdots + a_n \leq \sqrt{(n-j)(a_{j+1}^2 + \cdots + a_n^2)} < \sqrt{(n-1)(a_1^2 + \cdots + a_n^2)},\]

which is not possible. Therefore, all of \(a_1, \ldots, a_n\) are non-negative. \(\square\)

---

**Example 5: (Japan MO 2005)** If \(a, b, c\) are positive numbers with \(a + b + c = 1\), prove the inequality

\[a \sqrt{1+b-c} + b \sqrt{1+c-a} + c \sqrt{1+a-b} \leq 1.\]

**Solution:** Taking Holder’s inequality with \(p = 3/2\) and \(q = 3\), we have

\[\sum_{\text{cyc}} a \sqrt{1+b-c} = \sum_{\text{cyc}} a \frac{2}{3} (a(1+b-c))^\frac{3}{2} \leq \left( \sum_{\text{cyc}} a \right)^\frac{2}{3} \left( \sum_{\text{cyc}} (a(1+b-c))^\frac{3}{2} \right)^\frac{1}{3} = 1\]

Equality case holds when \(1+b-c = 1+c-a = 1+a-b\), which solves to \(a = b = c\). \(\square\)

(Note: This question is also solvable by using AM-GM inequality or Jensen’s inequality. Try it by yourself!)
Example 6: (BMO 1984) If \(a_1, a_2, \ldots, a_n\) \((n \geq 2)\) are positive real numbers with \(a_1 + a_2 + \ldots + a_n = 1\), prove that 
\[
\frac{a_1}{1 + a_2 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_2 + a_3 + \cdots + a_{n-1}} \geq \frac{n}{2n-1}
\]

Solution: The inequality above can be rewritten as \(\frac{a_1}{2 - a_1} + \cdots + \frac{a_n}{2 - a_n} \geq \frac{n}{2n-1}\).
WLOG, assume \(a_1 \geq \cdots \geq a_n\). Now we have \(\frac{1}{2 - a_1} \geq \cdots \geq \frac{1}{2 - a_n} > 0\). By Chebyshev’s sum inequality, we have
\[
\frac{1}{n} (a_1 + \cdots + a_n) \left( \frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n} \right) = \frac{1}{n} \left( \frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n} \right). 
\]
Since \(\frac{d^2(1/x)}{dx^2} = 2/x^3 > 0\) for all \(x > 0\), \(1/x\) is a convex function in \((0, +\infty)\). Applying Jensen’s inequality on \(1/x\), we have
\[
\frac{1}{n} \left( \frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n} \right) \geq \frac{1}{2n - a_1 - \cdots - a_n} = \frac{n}{2n - 1},
\]
which completes the proof. The equality case is \(a_1 = \cdots = a_n = 1/n\).

Example 7:
(IMO 1970 Longlisted (GDR)) Prove that for any triangle with sides \(a, b, c\) and area \(P\) the following inequality holds:
\[
P \leq \frac{\sqrt{3}}{4} (abc)^{2/3}.
\]
Find all triangles for which equality holds.

Solution: Using the formula \(P = \frac{1}{2}ab \sin C\), the given inequality is equivalent to
\[
\left( \frac{1}{2}ab \sin C \right)^{\frac{1}{3}} \left( \frac{1}{2}bc \sin A \right)^{\frac{1}{3}} \left( \frac{1}{2}ca \sin B \right)^{\frac{1}{3}} \leq \frac{\sqrt{3}}{4} (abc)^{2/3}
\]
\(\iff\) 
\[
(\sin A \sin B \sin C)^{\frac{1}{3}} \leq \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}
\]
Since \(\frac{d^2(\sin x)}{dx^2} = -\sin x \leq 0\) for all \(x \in [0, \pi]\), \(\sin x\) is concave. By AM-GM inequality and Jensen’s inequality, we have
\[
(\sin A \sin B \sin C)^{\frac{1}{3}} \leq \frac{\sin A + \sin B + \sin C}{3} \leq \sin \frac{A + B + C}{3} = \sin \frac{\pi}{3}.
\]
Equality case holds when \(A = B = C\), i.e. if and only if the triangle is equilateral.
Example 8: (IMO 1967 Longlisted (POL))

Prove that for arbitrary positive numbers the following inequality holds:

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}
\]

Solution 1: The given inequality can be rewritten as

\[
\frac{1}{2} \sum_{\text{sym}} a^0 b^0 c^{-1} \leq \frac{1}{2} \sum_{\text{sym}} a^5 b^{-3} c^{-3}.
\]

Since \((5, -3, -3) \succ (0, 0, -1)\), this is a result of Muirhead’s inequality.

Solution 2: The given inequality is equivalent to

\[
\frac{1}{2} \sum_{\text{sym}} a^3 b^3 c^2 \leq \frac{1}{2} \sum_{\text{sym}} a^8 b^0 c^0.
\]

Since \((8, 0, 0) \succ (3, 3, 2)\), this is a result of Muirhead’s inequality.

Example 9: (APMO 1996)

Let \(a, b, c\) be the lengths of the sides of a triangle. Prove that

\[
\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c},
\]

and determine when equality occurs.

Solution: \(\frac{d^{\sqrt{x}}}{dx} = -\frac{1}{4} x^{-3/2} < 0\) for all \(x > 0\), Therefore \(\sqrt{x}\) is concave. WLOG, assume \(a \geq b \geq c\). Now \((a + b - c, c + a - b, b + c - a) \succ (a, b, c)\). The given inequality is therefore a result of Karamata’s inequality. Equality holds when \(a = a + b - c\) and \(a + b = 2a\), which is equivalent to \(a = b = c\).

Example 10: (Poland 2005) Given \(a, b, c > 0\) and \(ab + bc + ca = 3\). Show that

\[
a^3 + b^3 + c^3 + 6abc \geq 9.
\]

Solution: By Maclaurin’s inequality, we have \(\frac{a+b+c}{3} \geq \sqrt[3]{\frac{ab+bc+ca}{3}} = 1\).

Now by Schur’s inequality, we have

\[
a^3 + b^3 + c^3 + 6abc \geq (a^2 b + a^2 c + b^2 a + b^2 c + c^2 a + c^2 b) + 3abc \\
= (a + b + c)(ab + bc + ca) \\
\geq 3 \cdot 3 = 9
\]  

Prepared by Leung W.C.
**Example 11:** (Turkevici inequality) For \( a, b, c, d \geq 0 \), prove the inequality
\[
a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.
\]
Solution: Let \( a = e^w, b = e^x, c = e^y \) and \( d = e^z \). Now the inequality is equivalent to
\[
e^{4w} + e^{4x} + e^{4y} + e^{4z} + 2e^{w+x+y+z} \geq e^{2a+2b} + e^{2a+2c} + e^{2a+2d} + e^{2b+2c} + e^{2b+2d} + e^{2c+2d}
\]
WLOG, assume \( w \geq x \geq y \geq z \)
If \( w + z \geq x + y \) we have
\[
\begin{align*}
4w & \geq 2w + 2x \\
4w + (w + x + y + z) & \geq 4w + (2x + 2y) = (2w + 2x) + (2w + 2y) \\
4w + 2(w + x + y + z) & = (2w + 2x) + (2w + 2y) + (2w + 2z)
\end{align*}
\]
Note: the majorization above is the result of the following calculations:
\[
4w \geq 2w + 2x \\
4x \geq 2x + 2y \\
4x + 4y \geq (2x + 2y) + (2x + 2z)
\]
If \( w + z < x + y \) we have
\[
\begin{align*}
(4w,4x,4y) & \succ (2w+2x,2w+2y,2x+2y) \\
(w+x+y+z, w+x+y+z, 4z) & \succ (2w+2z, 2x+2z, 2y+2z)
\end{align*}
\]
(The proof of the statements above is left to readers.)
Since \( \frac{d^2(e^x)}{dx^2} = e^x > 0 \), \( e^x \) is a convex function. Therefore, the given equation is true by applying Karamata’s inequality and adding up the results. Equality case holds when \( w = x = y = z \), i.e. \( a = b = c = d \).
\( \square \)
*(Note: To show that \( a \succ b \), the elements of \( b \) must be put in descending order.)*

**Notes on Inequalities**
Exercise:

1. Let $a, b, c > 0$. Prove that
\[ a + b + c \leq \frac{a^2 + b^2}{2c} + \frac{b^2 + c^2}{2a} + \frac{c^2 + a^2}{2b} \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}. \]

2. (IMO 1968 Shortlisted (POL))
If $a$ and $b$ are arbitrary positive real numbers and $m$ an integer, prove that
\[ \left(1 + \frac{a}{b}\right)^m + \left(1 + \frac{b}{a}\right)^m \geq 2^{m+1}. \]

3. Let $a_i, b_i, c_i, d_i$, where $i = 1, 2, \ldots, n$, be 4 sets of real numbers. Show that
\[
\left(\sum_{i=1}^{n} a_i^4\right)\left(\sum_{i=1}^{n} b_i^4\right)\left(\sum_{i=1}^{n} c_i^4\right)\left(\sum_{i=1}^{n} d_i^4\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i d_i\right)^4 \quad \text{and} \\
\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3.
\]

4. (Russia 1992) Let $x, y, z$ be positive numbers. Prove that $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$.

5. Let $n$ be a positive integer greater than 1. Show that $\sqrt{n(2^n - 1)} > \sum_{i=1}^{n} \sqrt{n}$.

6. Let $a_1, a_2, \ldots, a_n$ and $x_1, x_2, \ldots, x_n$ be two sets of positive real numbers. Show that
\[(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2 \leq (a_1 + a_2 + \cdots + a_n)(a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2).\]

7. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $S = a_1 + a_2 + \cdots + a_n$. Show that
\[
\frac{S}{S - a_1} + \frac{S}{S - a_2} + \cdots + \frac{S}{S - a_n} \geq \frac{n^2}{n - 1} \quad \text{and} \quad (1 + a_1)(1 + a_2) \cdots (1 + a_n) \leq \sum_{r=0}^{n} S^r.
\]

8. (IMO 1970 Longlisted (ROM))
If $a, b, c$ are side lengths of a triangle, prove that
\[(a + b)(b + c)(c + a) \geq 8(a + b - c)(b + c - a)(c + a - b).\]

9. (IMO 1970 Shortlisted (GDR))
Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be real numbers. Prove that
\[
1 + \sum_{i=1}^{n} (u_i + v_i)^2 \leq \frac{4}{3} \left(1 + \sum_{i=1}^{n} u_i^2\right) \left(1 + \sum_{i=1}^{n} v_i^2\right).
\]
In what case does equality hold?

10. Let $a, b, c > 0$. Prove that
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \leq \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c}.
\]

11. (Romania 2005, unused) Given $a, b, c > 0$, $a + b + c = 1$. Prove that
\[
\frac{a}{\sqrt{b + c}} + \frac{b}{\sqrt{c + a}} + \frac{c}{\sqrt{a + b}} \geq \frac{\sqrt{3}}{2}.
\]
12. Let $x, y, z$ be real numbers that $x + y + z = 0$. Prove that
\[ 6(x^3 + y^3 + z^3)^2 \leq (x^2 + y^2 + z^2)^3. \]

13. Let $a, b, x, y$ be positive real numbers, $c, z$ be real numbers, and $ac - b^2 = xz - y^2$. Show that $(a - x)(c - z) - (b - y)^2 \leq 0$.

14. (IMO 1983 #6) If $a, b,$ and $c$ are sides of a triangle, prove that
\[ a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0 \]
and determine when there is equality.

15. (Modified from IMO 1987 shortlist) Let $a, b, c > 0$, $m$ be a positive integer. Prove that
\[ \frac{a^m + b^m}{b + c} + \frac{b^m + c^m}{c + a} + \frac{c^m + a^m}{a + b} \geq \frac{3}{2} \left( \frac{a + b + c}{3} \right)^{m-1}. \]

16. (IMO 1966 Longlisted (YUG)) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove the inequality
\[ \left( \sum_{i=1}^{n} a_i \right)^2 \geq 4 \left( \sum_{i<j} a_i a_j \right)^2 \]
and find the conditions on the numbers $a_i$ for equality to hold.

17. (IMO 1975 #1) Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be two $n$-tuples of numbers. Prove that
\[ \sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2 \]
is true when $z_1, z_2, \ldots, z_n$ denote $y_1, y_2, \ldots, y_n$ taken in another order.

18. (IMO 1966 Longlisted (ROM)) If $n$ is a natural number, prove that
(a) $\log_{10}(n + 1) > \frac{3}{10n} + \log_{10} n$
(b) $\log n! > \frac{3n}{10} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 1 \right)$.

19. (Irish MO 1999) The sum of positive real numbers $a, b, c, d$ is 1. Prove that
\[ \frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a} \geq \frac{1}{2}, \]
with equality if and only if $a = b = c = d = \frac{1}{2}$.

20. (Belarus 1999) Given $a, b, c > 0, a^2 + b^2 + c^2 = 3$, prove that
\[ \frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}. \]
21. (IMO 1972 Shortlisted (CZS)) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1 + x_2 + \cdots + x_n = 0$. Let $m$ be the least and $M$ the greatest among them. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq -nmM.$$ 

22. (IMO 1984 #1) Let $x, y, z$ be nonnegative real numbers with $x + y + z = 1$. Show that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$ 

23. (IMO 1969 Longlisted (CZS)) Let $K_1, \ldots, K_n$ be nonnegative integers. Prove that

$$K_1!K_2! \cdots K_n! \geq \lfloor K/n \rfloor^n,$$

where $K = K_1 + \cdots + K_n$.

24. (IMO 1997 #3) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1 \quad \text{and} \quad |x_i| \leq \frac{n + 1}{2} \quad \text{for } i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, \ldots, y_n$ of the sequence $x_1, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n + 1}{2}.$$ 

25. (APMO 2007) Let $x, y$ and $z$ be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y + z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z + x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x + y)}} \geq 1.$$ 

26. (IMO 2005 #3) Let $x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$ 

27. (IMO 1969 #6) Under the conditions $x_1, x_2 > 0$, $x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}.$$ 

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