Notes on Inequalities

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Contents

1 Introduction to Inequalities

1.1 Law of trichotomy

For real numbers x and y, exactly one of the following holds: $x < y$, $x = y$, $x > y$.

1.2 Basic properties of inequalities

Here are the basic properties of inequalities, which are introduced in secondary schools:

Transitive property	If $a > b$, $b > c$, then $a > c$. Hence we write $a > b > c$.
Additive property	If $a > b$, then $a + c > b + c$ for all real number c.
Multiplicative property	If $a > b$, $c > 0$, then $ac > bc$. If $a > b$, $c < 0$, then $ac < bc$.
Reciprocal property	If $a > b > 0$ or $0 > a > b$, then $1/a < 1/b$.

1.3 Other elementary properties of inequalities

Besides the basic properties, here are a few other useful properties on inequalities:

(a) $a^2 > 0$ for all real number a;

(b) If $a > b$ and $c > d$, then $a + c > b + d$ and $a - d > b - c$;

(c) If $a > b > 0$ and $c > d > 0$, then $ac > bd$;

(d) If $a > b > 0$ and $0 > c > d$, then $ad < bc$;

(e) If $0 < a < 1 < b$ and $k > 0$, then $0 < a^k < 1 < a^{-k}$ and $0 < b^{-k} < 1 < b^k$;

(f) If $0 < a < b$ and $k > 0$, then $a^k < b^k$ and $a^{-k} > b^{-k}$.

Example: Show that $x^2 + y^2 \geq 2xy$, where x, y are real numbers.

Solution: $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$, which reduces to the given inequality. Equality case holds when $x = y$. $\square$

Example: Show that $(a + b)(b + c)(c + a) \geq 8abc$, where a, b, c are non-negative real numbers. Determine where equality case holds.

Solution 1: We have $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $c + a \geq 2\sqrt{ca}$. Multiplying all three inequality together, we get the required inequality. Equality case holds when $x = y = z$, $x = y = 0$, $y = z = 0$ or $z = x = 0$. $\square$

Solution 2: The inequality can be reduced into $a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 \geq 6abc$. This can be further reduced into $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0$, which is true. (For equality cases, see solution 1.) $\square$

Example: (IMO 1969 Longlisted (HUN)) Prove that $1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}<\frac{5}{4}$.

Solution: Note that $\frac{1}{n^3}<\frac{1}{n(n-1)(n+1)}=\frac{1}{2}\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)$. Therefore,

$$\begin{aligned}1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}&=1+\sum_{i=2}^{n}\frac{1}{i^3}<1+\sum_{i=2}^{n}\frac{1}{2}\left(\frac{1}{i-1}-\frac{2}{i}+\frac{1}{i+1}\right)\\&=1+\frac{1}{2}\left(\frac{1}{1}-\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}\right)<1+\frac{1}{2}\left(\frac{1}{1}-\frac{1}{2}\right)=\frac{5}{4}\end{aligned}$$ □

(Note: when n approaches infinity, the result is known as the Apéry's constant. Its value is approximately equal to 1.202.)

Exercise:

1. Show that $a^2+b^2+c^2\geq ab+bc+ca$, where a, b, c are real numbers. Determine when does equality hold.
2. Prove that $\sqrt{k+1}+\sqrt{k-1}<2\sqrt{k}$ and $\frac{1}{\sqrt{k}}<\sqrt{k+1}-\sqrt{k-1}$, where $k>1$.
3. Prove that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<\sqrt{n+1}+\sqrt{n-1}$ for any positive integer n.
4. Show that $\left(\frac{n+1}{2}\right)^n>n!$ for any integer $n>1$.
5. **(IMO 1970 Longlisted (FRA))** Let n and p be two integers such that $2p\leq n$. Prove the inequality
$$\frac{(n-p)!}{p!}\leq\left(\frac{n+1}{2}\right)^{n-2p}.$$
6. **(IMO 1975 Shortlisted (SWE))** Let M be the set of all positive integers that do not contain the digit 9 (base 10). If $x_1,\ldots,x_n$ are arbitrary but distinct elements in M, prove that
$$\sum_{j=1}^{n}\frac{1}{x_j}<80.$$
7. **(IMO 1982 #3)** Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties: $x_0=1$ and for all $i\geq 0$, $x_{i+1}\leq x_i$.
   (a) Prove that for every such sequence there is an $n\geq 1$ such that
   $$\frac{x_0^2}{x_1}+\frac{x_0^1}{x_2}+\cdots+\frac{x_{n-1}^2}{x_n}\geq 3.999.$$
   (b) Find such a sequence for which $\frac{x_0^2}{x_1}+\frac{x_0^1}{x_2}+\cdots+\frac{x_{n-1}^2}{x_n}<4$ for all n.

1.4 AM-GM inequality

For non-negative real numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

where the equality case happen if and only if $a_1 = \cdots = a_n$.

The expression $\frac{a_1+a_2+\cdots+a_n}{n}$ is known as the arithmetic mean (AM) of the n numbers, and the expression $\sqrt[n]{a_1 a_2 \cdots a_n}$ is known as the geometric mean (GM) of the n numbers.

Example: Given $a_1, \ldots, a_n \geq 0$. Show that $\sqrt[n]{a_1 a_2 \cdots a_n} \geq \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

Solution: By AM-GM inequality, we have $\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\frac{1}{a_2}\cdots\frac{1}{a_n}}$, which reduces to the required inequality. The equality case hold when $a_1 = \cdots = a_n$. □

Note: the right hand side of the inequality is known as the harmonic mean (HM).

Example:
Show that if a, b, c are positive real numbers, then $a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Solution: By AM-GM inequality, we have

$$\begin{aligned} 2a^4 + b^4 + c^4 &\geq 4a^2bc \\ a^4 + 2b^4 + c^4 &\geq 4ab^2c \\ a^4 + b^4 + 2c^4 &\geq 4abc^2 \end{aligned}$$

Adding up becomes $4(a^4 + b^4 + c^4) \geq 4abc(a + b + c)$, which reduces to the required inequality. Equality case holds when $a = b = c$. □

1.5 Sides of a triangle

Line segments with lengths a, b, c can form a triangle if and only if $a + b > c$, $b + c > a$ and $c + a > b$. This is known as the triangle inequality.

Also, if a, b, c are the lengths of the sides of a triangle, we can make the substitution $x = \frac{c+a-b}{2}$, $y = \frac{a+b-c}{2}$ and $z = \frac{b+c-a}{2}$. By triangle inequality, we have $x, y, z > 0$. Solving for a, b, and c, we obtain $a = x + y$, $b = y + z$, $c = z + x$ and $a + b + c = 2(x + y + z)$.

Of course, other formulas related to triangles can also be useful, i.e. $S = \frac{1}{2}ab \sin C$, cosine formula ($c^2 = a^2 + b^2 - 2ab \cos C$) and Heron's formula ($S = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$).

Example: (IMO 1964 #2) Let a, b, c be the lengths of the sides of a triangle. Prove that $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\le 3abc$.

Solution: Let $x=\frac{c+a-b}{2}$, $y=\frac{a+b-c}{2}$ and $z=\frac{b+c-a}{2}$. We have $a=x+y$, $b=y+z$, $c=z+x$. Now the given inequality is equivalent to

$$
\begin{aligned}
& (x+y)^2(2z)+(y+z)^2(2x)+(z+x)^2(2y)\le 3(x+y)(y+z)(z+x) \\
\iff\quad & 2\begin{pmatrix} x^2y+xy^2+y^2z+ \\ yz^2+z^2x+zx^2\end{pmatrix}+12xyz\le 3\begin{pmatrix} x^2y+xy^2+y^2z+ \\ yz^2+z^2x+zx^2\end{pmatrix}+6xyz \\
\iff\quad & 6xyz\le x^2y+xy^2+y^2z+yz^2+z^2x+zx^2
\end{aligned}
$$

Which is a direct result of AM-GM inequality. Since $x,y,z>0$, the equality case of the above inequality is $x=y=z$, i.e. $a=b=c$. $\square$

Exercise:

1. Given $x,y,z\ge 0$. Prove that $x^2y^2+y^2z^2+z^2x^2\ge xyz(x+y+z)$. (Does the equality hold also for all real numbers x, y and z?)
2. Let a, b, c be the lengths of the sides of a triangle. Show that $(a+b+c)(b+c-a)<4bc$.
3. Let a, b, c be real numbers. If $a+b+c=1$, prove that $a^2+b^2+c^2\ge\dfrac{1}{3}$.
4. Given $-1<x,y,z<1$. Prove that
$$\frac{1}{(1-x)(1-y)(1-z)}+\frac{1}{(1+x)(1+y)(1+z)}\ge 2.$$
5. **(BMO 2001)** Let a, b, c be positive real numbers such that $a+b+c\ge abc$. Prove that $a^2+b^2+c^2\ge\sqrt{3}abc$.
6. **(Yugoslavia 1987)** Given $a,b>0$. Prove that $\dfrac{1}{2}(a+b)^2+\dfrac{1}{4}(a+b)\ge a\sqrt{b}+b\sqrt{a}$.
7. **(Nesbitt's inequality)** Given $a,b,c>0$. Prove that $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\ge\dfrac{3}{2}$.
8. **(IMO 1969 Longlisted (YUG))**
Suppose that positive real numbers x_1,x_2,x_3 satisfy
$$x_1x_2x_3>1,\qquad x_1+x_2+x_3<\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}.$$
Prove that:
    (a) None of x_1, x_2, x_3 equals 1.
    (b) Exactly one of these numbers is less than 1.

2 Classical inequalities

2.1 Mean inequalities

For **positive real numbers** $a_1, \ldots, a_n$, we have $QM \geq AM \geq GM \geq HM$, where

$$QM = \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}, \quad AM = \frac{a_1 + \cdots + a_n}{n},$$
$$GM = \sqrt[n]{a_1 a_2 \cdots a_n}, \quad HM = \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}},$$

The equality case is $a_1 = \cdots = a_n$. QM, AM, GM and HM stands for quadratic mean, arithmetic mean, geometric mean and harmonic mean respectively. If HM is not involved, the conditions can be relaxed into $a_1, \ldots, a_n \geq 0$.

2.2 Generalized mean inequality (or power mean inequality)

For **positive real numbers** $a_1, \ldots, a_n$, we define the generalized mean or power mean with exponent p as follows:

$$M_p = \begin{cases} \left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} & \text{for } p \neq 0 \\ \displaystyle\lim_{p \to 0}\left(\dfrac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} = \sqrt[n]{a_1 a_2 \cdots a_n} & \text{for } p = 0 \end{cases}$$

If $p > q$, then we have $M_p \geq M_q$, with the equality case $a_1 = \cdots = a_n$.

Note the following special cases: $M_{+\infty} = \max(a_1, \ldots, a_n)$, $M_2 = QM$, $M_1 = AM$, $M_0 = GM$, $M_{-1} = HM$, $M_{-\infty} = \min(a_1, \ldots, a_n)$.

2.3 Jensen's inequality

For a convex function f, numbers $x_1, x_2, \ldots, x_n$ in its domain, and positive weights $a_1, \ldots, a_n$ that $a_1 + \cdots + a_n = 1$, Jensen's inequality can be stated as

$$f(\sum_{i=1}^{n} a_i x_i) \leq \sum_{i=1}^{n} a_i f(x_i).$$

If the weights are equal, then we have $f(\displaystyle\sum_{i=1}^{n} \frac{x_i}{n}) \leq \frac{\sum_{i=1}^{n} f(x_i)}{n}$.

For a concave function, the inequality sign is reversed.

Note: function f is convex if, for any points x_1, x_2 is in the domain of f, $t \in (0,1)$, we have $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ provided that $f(tx_1 + (1-t)x_2)$ is defined. This is the two variable case of Jensen's inequality.

2.4 Cauchy–Schwarz inequality

For real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, we have

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Equality holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ or $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$.

2.5 Hölder's inequality

Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be real numbers. If $p, q > 0$, $1/p + 1/q = 1$, we have

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}.$$

Equality holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ or $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$. The CS inequality is a special case of Hölder's inequality where $p = q = 2$.

2.6 Rearrangement inequality

Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have the following:

$$\sum_{i=1}^n x_i y_i \geq \sum_{i=1}^n x_i y_{\sigma_i} \geq \sum_{i=1}^n x_i y_{n+1-i}$$

2.7 Chebeshev's sum inequality

Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have the following:

$$\sum_{i=1}^n x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) \geq \sum_{i=1}^n x_i y_{n+1-i}$$

2.8 Minkowski's inequality

For **non-negative real numbers** $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, $p \geq 1$, we have

$$\left(\sum_{k=1}^n (x_k + y_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^n x_k^p\right)^{1/p} + \left(\sum_{k=1}^n y_k^p\right)^{1/p}.$$

Equality case holds if and only if $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$, $\frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n}$, or $p = 1$.

When $p = 2$, then we obtain the triangle inequality in the n-dimensional space:

$$\sqrt{\sum_{k=1}^n (x_k + y_k)^2} \leq \sqrt{\sum_{k=1}^n x_k^2} + \sqrt{\sum_{k=1}^n y_k^2}$$

2.9 Majorization of finite sequences

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two sequences of real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Sequence a majorizes sequence b if the following two conditions are satisfied:

(i) $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k$, for all k where $1 \leq k \leq n - 1$;
(ii) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$.

We write $a \succ b$ or $b \prec a$ if a majorizes b.

2.10 Karamata's inequality

For a convex function f, given $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$, we have

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

The inequality sign is reversed if the function is concave.

Karamata's inequality is also known as majorization inequality. Note that it is a generalization of Jensen's inequality.

2.11 Cyclic and symmetric summation

For inequalities we often have do summation in cyclic and symmetric way. To avoid writing a lot of terms, we use the symbol $\sum_{\text{cyc}}$ for cyclic sums, and $\sum_{\text{sym}}$ for symmetric sums, i.e. summation of all terms in the permutation. For example:

$$\sum_{\text{cyc}(x,y,z)} x^2y = x^2y + y^2z + z^2x$$
$$\sum_{\text{sym}(x,y,z)} x^2y = x^2y + y^2z + z^2x + y^2x + z^2y + x^2z$$

$\sum_{\text{cyc}(x,y,z)} x^2y$ can be notated $\sum_{x,y,z} x^2y$, while $\sum_{\text{sym}(x,y,z)} x^2y$ can be notated $\sum_{\text{sym}} x^2y^1z^0$.

2.12 Muirhead's inequality

Let $x_1, \ldots, x_n$ be **non-negative integers**. If $\alpha = (\alpha_1, \ldots, \alpha_n) \succ \beta = (\beta_1, \ldots, \beta_n)$, then

$$\sum_{\text{sym}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq \sum_{\text{sym}} x_1^{\beta_1} \cdots x_n^{\beta_n}$$

Equality case holds if $\alpha = \beta$ or $x_1 = \cdots = x_m$. Conversely, if the inequality holds for all non-negative $x_1, \ldots, x_n$, then $\alpha \succ \beta$.

2.13 Schur's inequality

For positive numbers x, y, z and real number t, we have

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0$$

Equality holds if $x = y = z$ or two of them are equal and the third is zero. If t is a positive even number, then the inequality holds for all real numbers x, y, z. If t is positve, then the inequality holds for all non-negative numbers x, y, z.

Schur's inequality can be used to form many difficult to prove inequalities.

When $r = 1$, we have:

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2$$
$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

When $r = 2$, we have:

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3$$

2.14 Elementary symmetric polynomial

Consider the coefficients e_i of polynomial $(t+x_1)\cdots(t+x_n) = t^n + e_1t^{n-1} + \cdots + e_{n-1}t + e_n$. For example:

$$\begin{aligned} e_1(x, y, z) &= x + y + z \\ e_2(x, y, z) &= xy + yz + zx \\ e_3(x, y, z) &= xyz \end{aligned}$$

The coefficients $e_1, \ldots, e_n$ are known as the elementary symmetric polynomials.

2.15 Symmetric mean inequalities

We define $S_i = e_i/\binom{n}{i}$. S_i are known as the symmetric means. Note that S_1 is the arithmetic mean, and S_n is the geometric mean. We have the following inequalities:

Newton's inequality: $S_i^2 \geq S_{i+1}S_{i-1}$, and

Maclaurin's inequality: $S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n}$.

2.16 Bernoulli's inequality

For all integer $r \geq 1$ and $x \geq -1$, we have $(1+x)^r \geq 1 + rx$. It can be generalized to real exponents: for $x > -1, x \neq 0$, we have

$$\begin{cases} (1+x)^a > 1 + ax & \text{for } a > 1 \text{ or } a < 0 \\ (1+x)^a < 1 + ax & \text{for } 0 < a < 1 \end{cases}.$$

Note that equality holds if $x = 0$.

3 Examples

Example 1: (IMO 1970 Longlisted (AUT)) Prove that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\le\frac{1}{2}(a+b+c)\quad(a,b,c>0).$$

Solution: By GM-HM inequality,

$$\begin{aligned}\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}&=\frac{1}{\frac{1}{b}+\frac{1}{c}}+\frac{1}{\frac{1}{c}+\frac{1}{a}}+\frac{1}{\frac{1}{a}+\frac{1}{b}}\\&<\frac{1}{2}\left(\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}\right)\\&=\frac{1}{2}(a+b+c).\end{aligned}$$

Equality case holds for $a=b=c$. □

Example 2: Let $a,b,c\ge 0$. Prove that $\dfrac{a^8}{bc}+\dfrac{b^8}{ca}+\dfrac{c^8}{ab}\ge a^6+b^6+c^6$.

Solution 1: We have $(8,-1,-1)\succ(6,0,0)$. Therefore the given inequality is a result of Muirhead's inequality. Equality case holds when $a=b=c$. □

(Note: The given inequality can be rewritten as $\frac{1}{2}\sum_{\text{sym}}a^8b^{-1}c^{-1}\ge\frac{1}{2}\sum_{\text{sym}}a^6b^0c^0$.)

Solution 2: The given inequality is equivalent to $a^9+b^9+c^9\ge abc(a^6+b^6+c^6)$. WLOG, assume $a\ge b\ge c$. By Chebyshev's sum inequality and AM-GM inequality, we have

$$a^9+b^9+c^9\ge\frac{1}{3}(a^3+b^3+c^3)(a^6+b^6+c^6)\ge abc(a^6+b^6+c^6),$$

which completes the proof. Equality case holds when $a=b=c$. □

Example 3: Let $a,b,c\ge 0$. Prove that $a^5+b^5+c^5\ge a^4b+b^4c+c^4a$.

Solution: If $a\ge b\ge c$, we have $a^4\ge b^4\ge c^4$. If a, b and c has a different order, we have a similar result. By rearrangement inequality, we have

$$a^5+b^5+c^5=a^4\cdot a+b^4\cdot b+c^4\cdot c\ge a^4b+b^4c+c^4a.$$

Equality case holds when $a=b=c$. □

(Note: avoid using the phrase "WLOG, assume $a\ge b\ge c$" in this question because the expression $a^4b+b^4c+c^4a$ is not symmetric.)

Example 4: (IMO 1966 Longlisted (CZS))

(a) Prove that $(a_1 + a_2 + \cdots + a_k)^2 \leq k(a_1^2 + \cdots + a_k^2)$, where $k \geq 1$ is a natural number and $a_1, \ldots, a_k$ are arbitrary real numbers.

(b) If real numbers $a_1, \ldots, a_n$ satisfy

$$a_1 + a_2 + \cdots + a_n \geq \sqrt{(n-1)(a_1^2 + \cdots + a_n^2)},$$

show that they are all nonnegative.

Solution:

(a) Let $b_1 = \cdots = b_k = 1$. We have

$$(a_1 b_1 + a_2 b_2 + \cdots + a_k b_2)^2 \leq (a_1^2 + \cdots + a_k^2)(b_1^2 + \cdots + b_k^2),$$

which reduces to the required inequality. Equality case holds when $a_1 = \cdots = a_k$.

(b) WLOG, assume that $a_1, \ldots, a_j$ are negative, where $1 \leq j \leq n$. Now

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &< a_{j+1} + \cdots + a_n \\ &\leq \sqrt{(n-j)(a_{j+1}^2 + \ldots + a_n^2)} \\ &< \sqrt{(n-1)(a_1^2 + \ldots + a_n^2)}, \end{aligned}$$

which is not possible. Therefore, all of $a_1, \ldots, a_n$ are non-negative. $\square$

Example 5: (Japan MO 2005) If a, b, c are positive numbers with $a + b + c = 1$, prove the inequality

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq 1.$$

Solution: Taking Holder's inequality with $p = 3/2$ and $q = 3$, we have

$$\sum_{\text{cyc}} a\sqrt[3]{1+b-c} = \sum_{\text{cyc}} a^{\frac{2}{3}}(a(1+b-c))^{\frac{1}{3}}$$

$$\leq \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}} \left(\sum_{\text{cyc}} a(1+b-c)\right)^{\frac{1}{3}} = \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}} \left(\sum_{\text{cyc}} a\right)^{\frac{1}{3}} = 1$$

Equality case holds when $1+b-c = 1+c-a = 1+a-b$, which solves to $a = b = c$. $\square$

(Note: This question is also solvable by using AM-GM inequality or Jensen's inequality. Try it by yourself!)

Example 6: (BMO 1984) If $a_1, a_2, \ldots, a_n$ ($n \geq 2$) are positive real numbers with $a_1 + a_2 + \ldots + a_n = 1$, prove that

$$\frac{a_1}{1 + a_2 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_2 + a_3 + \cdots + a_{n-1}} \geq \frac{n}{2n-1}$$

Solution: The inequality above can be rewritten as $\dfrac{a_1}{2 - a_1} + \cdots + \dfrac{a_n}{2 - a_n} \geq \dfrac{n}{2n-1}$. WLOG, assume $a_1 \geq \cdots \geq a_n$. Now we have $\frac{1}{2-a_1} \geq \ldots \geq \frac{1}{2-a_n} > 0$. By Chebyshev's sum inequality, we have

$$\begin{aligned} &\frac{a_1}{1 + a_2 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_2 + a_3 + \cdots + a_{n-1}} \\ \geq &\frac{1}{n}(a_1 + \cdots + a_n)\left(\frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n}\right) = \frac{1}{n}\left(\frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n}\right). \end{aligned}$$

Since $\frac{d^2(1/x)}{dx^2} = 2/x^3 > 0$ for all $x > 0$, $1/x$ is a convex function in $(0, +\infty)$. Applying Jensen's inequality on $1/x$, we have

$$\frac{1}{n}\left(\frac{1}{2 - a_1} + \cdots + \frac{1}{2 - a_n}\right) \geq \frac{1}{(2n - a_1 - \cdots - a_n)/n} = \frac{n}{2n-1},$$

which completes the proof. The equality case is $a_1 = \cdots = a_n = 1/n$. $\square$

Example 7:

(IMO 1970 Longlisted (GDR)) Prove that for any triangle with sides a, b, c and area P the following inequality holds:

$$P \leq \frac{\sqrt{3}}{4}(abc)^{2/3}.$$

Find all triangles for which equality holds.

Solution: Using the formula $P = \frac{1}{2}ab\sin C$, the given inequality is equivalent to

$$\begin{aligned} \left(\frac{1}{2}ab\sin C\right)^{\frac{1}{3}}\left(\frac{1}{2}bc\sin A\right)^{\frac{1}{3}}\left(\frac{1}{2}ca\sin B\right)^{\frac{1}{3}} &\leq \frac{\sqrt{3}}{4}(abc)^{2/3} \\ \iff \qquad (\sin A \sin B \sin C)^{\frac{1}{3}} &\leq \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \end{aligned}$$

Since $\dfrac{d^2(\sin x)}{dx^2} = -\sin x \leq 0$ for all $x \in [0, \pi]$, $\sin x$ is concave. By AM-GM inequality and Jensen's inequality, we have

$$(\sin A \sin B \sin C)^{\frac{1}{3}} \leq \frac{\sin A + \sin B + \sin C}{3} \leq \sin\frac{A + B + C}{3} = \sin\frac{\pi}{3}.$$

Equality case holds when $A = B = C$, i.e. if and only if the triangle is equilateral. $\square$

Example 8: (IMO 1967 Longlisted (POL))

Prove that for arbitrary positive numbers the following inequality holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{a^8+b^8+c^8}{a^3b^3c^3}$$

Solution 1: The given inequality can be rewritten as $\frac{1}{2}\sum_{\text{sym}}a^0b^0c^{-1}\leq\frac{1}{2}\sum_{\text{sym}}a^5b^{-3}c^{-3}$. Since $(5,-3,-3)\succ(0,0,-1)$, this is a result of Muirhead's inequality. □

Solution 2: The given inequality is equivalent to $a^2b^3c^3+a^3b^2c^3+a^3b^3c^2\leq a^8+b^8+c^8$, i.e. $\frac{1}{2}\sum_{\text{sym}}a^3b^3c^2\leq\frac{1}{2}\sum_{\text{sym}}a^8b^0c^0$. Since $(8,0,0)\succ(3,3,2)$, this is a result of Muirhead's inequality. □

Example 9: (APMO 1996)

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b}\leq\sqrt{a}+\sqrt{b}+\sqrt{c},$$

and determine when equality occurs.

Solution: $\frac{d^2\sqrt{x}}{dx^2}=-\frac{1}{4}x^{-3/2}<0$ for all $x>0$, Therefore $\sqrt{x}$ is concave. WLOG, assume $a\geq b\geq c$. Now $(a+b-c,c+a-b,b+c-a)\succ(a,b,c)$. The given inequality is therefore a result of Karamata's inequality. Equality holds when $a=a+b-c$ and $a+b=2a$, which is equivalent to $a=b=c$. □

Example 10: (Poland 2005) Given $a,b,c>0$ and $ab+bc+ca=3$. Show that

$$a^3+b^3+c^3+6abc\geq 9.$$

Solution: By Maclaurin's inequality, we have $\frac{a+b+c}{3}\geq\sqrt{\frac{ab+bc+ca}{3}}=1$.

Now by Schur's inequality, we have

$$\begin{aligned}a^3+b^3+c^3+6abc&\geq(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+3abc\\&=(a+b+c)(ab+bc+ca)\\&\geq 3\cdot 3=9\end{aligned}$$

□

Example 11: (Turkevici inequality) For $a, b, c, d \geq 0$, prove the inequality

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

Solution: Let $a = e^w, b = e^x, c = e^y$ and $d = e^z$. Now the inequality is equivalent to

$$e^{4w} + e^{4x} + e^{4y} + e^{4z} + 2e^{w+x+y+z} \geq e^{2a+2b} + e^{2a+2c} + e^{2a+2d} + e^{2b+2c} + e^{2b+2d} + e^{2c+2d}$$

WLOG, assume $w \geq x \geq y \geq z$

If $w + z \geq x + y$ we have

$$\begin{cases} (4w, w+x+y+z, w+x+y+z) \succ (2w+2x, 2w+2y, 2w+2z) \\ (4x, 4y, 4z) \succ (2x+2y, 2x+2z, 2y+2z) \end{cases}$$

Note: the majorization above is the result of the following calculations:

$$\begin{aligned} 4w &\geq 2w + 2x \\ 4w + (w+x+y+z) &\geq 4w + (2x+2y) = (2w+2x) + (2w+2y) \\ 4w + 2(w+x+y+z) &= (2w+2x) + (2w+2y) + (2w+2z) \end{aligned}$$

$$\begin{aligned} 4x &\geq 2x + 2y \\ 4x + 4y &\geq (2x+2y) + (2x+2z) \\ 4x + 4y + 4z &= (2x+2y) + (2x+2z) + (2y+2z) \end{aligned}$$

If $w + z < x + y$ we have

$$\begin{cases} (4w, 4x, 4y) \succ (2w+2x, 2w+2y, 2x+2y) \\ (w+x+y+z, w+x+y+z, 4z) \succ (2w+2z, 2x+2z, 2y+2z) \end{cases}$$

(The proof of the statements above is left to readers.)

Since $\frac{d^2(e^x)}{dx^2} = e^x > 0$, e^x is a convex function. Therefore, the given equation is true by applying Karamata's inequality and adding up the results. Equality case holds when $w = x = y = z$, i.e. $a = b = c = d$. $\square$

(Note: To show that $a \succ b$, the elements of b must be put in descending order.)

Exercise:

1. Let $a, b, c > 0$. Prove that
$$a+b+c \leq \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b} \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}.$$

2. **(IMO 1968 Shortlisted (POL))**
If a and b are arbitrary positive real numbers and m an integer, prove that
$$\left(1+\frac{a}{b}\right)^m + \left(1+\frac{b}{a}\right)^m \geq 2^{m+1}.$$

3. Let a_i, b_i, c_i, d_i, where $i = 1, 2, \ldots, n$, be 4 sets of real numbers. Show that $\left(\sum_{i=1}^n a_i^4\right)\left(\sum_{i=1}^n b_i^4\right)\left(\sum_{i=1}^n c_i^4\right)\left(\sum_{i=1}^n d_i^4\right) \geq \left(\sum_{i=1}^n a_i b_i c_i d_i\right)^4$ and $\left(\sum_{i=1}^n a_i^3\right)\left(\sum_{i=1}^n b_i^3\right)\left(\sum_{i=1}^n c_i^3\right) \geq \left(\sum_{i=1}^n a_i b_i c_i\right)^3$.

4. **(Russia 1992)** Let x, y, z be positive numbers. Prove that $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$.

5. Let n be a positive integer greater than 1. Show that $\sqrt{n(2^n-1)} > \sum_{i=1}^n \sqrt{\binom{n}{i}}$.

6. Let $a_1, a_2, \ldots, a_n$ and $x_1, x_2, \ldots, x_n$ be two sets of positive real numbers. Show that $(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \leq (a_1 + a_2 + \cdots + a_n)(a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2)$.

7. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $S = a_1 + a_2 + \cdots + a_n$. Show that $\frac{S}{S-a_1} + \frac{S}{S-a_2} + \cdots + \frac{S}{S-a_n} \geq \frac{n^2}{n-1}$ and $(1+a_1)(1+a_2)\cdots(1+a_n) \leq \sum_{r=0}^n \frac{S^r}{r!}$.

8. **(IMO 1970 Longlisted (ROM))**
If a, b, c are side lengths of a triangle, prove that
$$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b).$$

9. **(IMO 1970 Shortlisted (GDR))**
Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be real numbers. Prove that
$$1 + \sum_{i=1}^n (u_i+v_i)^2 \leq \frac{4}{3}\left(1+\sum_{i=1}^n u_i^2\right)\left(1+\sum_{i=1}^n v_i^2\right).$$
In what case does equality hold?

10. Let $a, b, c > 0$. Prove that $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c}$.

11. **(Romania 2005, unused)** Given $a, b, c > 0$, $a+b+c = 1$. Prove that
$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

12. Let x, y, z be real numbers that $x+y+z=0$. Prove that
$$6(x^3+y^3+z^3)^2 \leq (x^2+y^2+z^2)^3.$$

13. Let a, b, x, y be positive real numbers, c, z be real numbers, and $ac-b^2 = xz-y^2$. Show that $(a-x)(c-z)-(b-y)^2 \leq 0$.

14. **(IMO 1983 #6)** If a, b, and c are sides of a triangle, prove that
$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0$$
and determine when there is equality.

15. **(Modified from IMO 1987 shortlist)**
Let $a, b, c > 0$, m be a positive integer. Prove that
$$\frac{a^m}{b+c}+\frac{b^m}{c+a}+\frac{c^m}{a+b} \geq \frac{3}{2}\left(\frac{a+b+c}{3}\right)^{m-1}$$

16. **(IMO 1966 Longlisted (YUG))**
Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove the inequality
$$\binom{n}{2}\sum_{i<j}\frac{1}{a_ia_j} \geq 4\left(\sum_{i<j}\frac{1}{a_i+a_j}\right)^2$$
and find the conditions on the numbers a_i for equality to hold.

17. **(IMO 1975 #1)** Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be two n-tuples of numbers. Prove that
$$\sum_{i=1}^n (x_i-y_i)^2 \leq \sum_{i=1}^n (x_i-z_i)^2$$
is true when $z_1, z_2, \ldots, z_n$ denote $y_1, y_2, \ldots, y_n$ taken in another order.

18. **(IMO 1966 Longlisted (ROM))** If n is a natural number, prove that
    (a) $\log_{10}(n+1) > \frac{3}{10n}+\log_{10} n$
    (b) $\log n! > \frac{3n}{10}(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-1)$.

19. **(Irish MO 1999)** The sum of positive real numbers a, b, c, d is 1. Prove that
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a} \geq \frac{1}{2},$$
with equality if and only if $a=b=c=d=\frac{1}{4}$.

20. **(Belarus 1999)** Given $a, b, c > 0$, $a^2+b^2+c^2 = 3$, prove that
$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca} \geq \frac{3}{2}.$$

21. **(IMO 1972 Shortlisted (CZS))** Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1 + x_2 + \cdots + x_n = 0$. Let m be the least and M the greatest among them. Prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq -nmM.$$

22. **(IMO 1984 #1)**
Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Show that
$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

23. **(IMO 1969 Longlisted (CZS))**
Let $K_1, \ldots, K_n$ be nonnegative integers. Prove that
$$K_1!K_2! \cdots K_n! \geq [K/n]!^n,$$
where $K = K_1 + \cdots + K_n$.

24. **(IMO 1997 #3)** Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions
$$|x_1 + x_2 + \cdots + x_n| = 1 \quad \text{and} \quad |x_i| \leq \frac{n+1}{2} \quad \text{for } i = 1, 2, ..., n.$$
Show that there exists a permutation $y_1, \ldots, y_n$ of the sequence $x_1, \ldots, x_n$ such that
$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

25. **(APMO 2007)** Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$. Prove that
$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}} + \frac{y^2+zx}{\sqrt{2y^2(z+x)}} + \frac{z^2+xy}{\sqrt{2z^2(x+y)}} \geq 1.$$

26. **(IMO 2005 #3)** Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that
$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{x^2+y^5+z^2} + \frac{z^5-z^2}{x^2+y^2+z^5} \geq 0.$$

27. **(IMO 1969 #6)**
Under the conditions $x_1, x_2 > 0$, $x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove the inequality
$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}.$$